

Classification of Quiver Hopf Algebras and Pointed Hopf Algebras of Type One

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Abstract

The quiver Hopf algebras are classified by means of ramification system with irreducible representations. This leads to the classification of Nichols algebras over group algebras and pointed Hopf algebras of type one.

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0 Introduction

Hopf algebras have important applications in mathematics and mathematical physics. Indeed, quasitriangular Hopf algebras gives rise to braided tensor categories through their categories of representations (see [ENO, EO]). Semisimple Hopf algebras and non-semisimple Hopf algebras are related to conformal field theories (see [Ga]).

The classification of Hopf algebras is the main object in research of Hopf algebras. So far, much important results have been obtained in the classification of finite dimensional pointed Hopf algebras (see [AS98, AS02, AS00, AS05, H1, H2, AZ07]). The classification of PM quiver Hopf algebras was completed by means of ramification system with characters (RSC in short) in [ZZC]. The classification of RSC's over symmetric group \mathbb{S}_n with $n \neq 6$ was obtained in [ZWW]. Irreducible Hopf bimodules over a finite group were described in [DPR]. They correspond to pairs (\mathcal{O}_s, ρ) , where \mathcal{O}_s is a conjugacy class containing s in

G and ρ is an irreducible representation of the centralizer G^s in G . This result has been reobtained and reproved several times; these data are used in a previous paper of Lusztig more or less in the same direction as in [DPR]. The relation between bi-one arrow Nichols algebras and $\mathfrak{B}(\mathcal{O}_s, \rho)$, which is the Nichols algebra corresponding to pairs (\mathcal{O}_s, ρ) , was given in [ZZWC].

In this paper, quiver Hopf algebras, Nichols algebras over group algebras and pointed Hopf algebras of type one are classified by means of ramification system with irreducible representations (RSR in short). As examples we classify RSR's over symmetric group \mathbb{S}_n with $n \neq 6$.

The quivers (see [ARS, CR02, CR97, ZZ, OZ] Ri84) and the tensor algebras of Hopf bimodule (see [N, W]) have widely been applied in representation theory, Hopf algebras and quantum groups. We use the quivers to describe Yetter-Drinfeld kG -modules, kG -Hopf bimodules, Nichols algebras in braided tensor category ${}^{kG}_{kG}\mathcal{YD}$, pointed Hopf algebras of type one and quiver Hopf algebras in this paper.

The main results in this paper are summarized in the following statement.

Theorem 1. (*Classification theorem*) (i) *Every Yetter-Drinfeld kG -module*, (ii) *every kG -Hopf bimodule*, (iii) *every Nichols algebra in braided tensor category ${}^{kG}_{kG}\mathcal{YD}$* , (iv) *every pointed Hopf algebra of type one with coradical kG and* (v) *every quiver Hopf algebra over G are, respectively, isomorphic to one of the following: $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$, $(kQ_1^c, G, r, \vec{\rho}, u)$, $\mathcal{B}(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$, $kG[kQ_1^c, G, r, \vec{\rho}, u]$, $kQ^c(G, r, \vec{\rho}, u)$, $kQ^s(G, r, \vec{\rho}, u)$, $kG[kQ_1^c, G, r, \vec{\rho}, u]$, $kQ^a(G, r, \vec{\rho}, u)$, $kQ^{sc}(G, r, \vec{\rho}, u)$, $(kG)^*[kQ_1^a, G, r, \vec{\rho}, u]$. Furthermore, they are uniquely determined by $\text{RSR}(G, r, \vec{\rho}, u)$ up to isomorphisms of RSR's.*

Indeed, (i) follows from Proposition 2.4(i); (ii) follows from Proposition 1.3; (iii) follows from Proposition 2.4(ii); (iv) follows from Theorem 3; (v) follows from definition of quiver Hopf algebras in [ZZC, the argument after Theorem 2]. For the uniqueness, (i) and (iii) follow from Theorem 4; (ii), (iv) and (v) follow from Theorem 2.

Note that the isomorphism in (i) is about Yetter-Drinfeld kG -modules; the isomorphism in (ii) is about kG -Hopf bimodules; the isomorphism in (iii) is about graded braided Hopf algebras in ${}^{kG}_{kG}\mathcal{YD}$; the isomorphism in (iv) and (v) are about graded Hopf algebras.

Furthermore, for any fixed map u_0 from $\mathcal{K}(G)$ to G with $u_0(C) \in C$ for any $C \in \mathcal{K}(G)$, if u_0 replaces u in the theorem above, then the theorem holds still but the isomorphisms in (i), (ii) and (iii) need change into graded pull-push isomorphisms.

Preliminaries

Throughout this paper we assume that G is a finite group and k is a field with $\text{char } k \nmid |G|$.

Let \hat{G} denote the set of all isomorphic classes of irreducible representations of group G and Z_s the centralizer of s in G . For $h \in G$ and an isomorphism ϕ from G to G' , define a map ϕ_h from G to G' by sending x to $\phi(h^{-1}xh)$ for any $x \in G$. $\deg(\rho)$ denotes the dimension of representation space of representation ρ .

Let \mathbb{N} and \mathbb{Z} denote the sets of all non-negative integers and all integers, respectively. For a set X , we denote by $|X|$ the number of elements in X . If $X = \oplus_{i \in I} X_{(i)}$ as vector spaces, then we denote by ι_i the natural injection from $X_{(i)}$ to X and by π_i the corresponding projection from X to $X_{(i)}$. We will use μ to denote the multiplication of an algebra and use Δ to denote the comultiplication of a coalgebra. For a (left or right) module and a (left or right) comodule, denote by α^- , α^+ , δ^- and δ^+ the left module, right module, left comodule and right comodule structure maps, respectively. The Sweedler's sigma notations for coalgebras and comodules are $\Delta(x) = \sum x_{(1)} \otimes x_{(2)}$, $\delta^-(x) = \sum x_{(-1)} \otimes x_{(0)}$, $\delta^+(x) = \sum x_{(0)} \otimes x_{(1)}$.

A quiver $Q = (Q_0, Q_1, s, t)$ is an oriented graph, where Q_0 and Q_1 are the sets of vertices and arrows, respectively; s and t are two maps from Q_1 to Q_0 . For any arrow $a \in Q_1$, $s(a)$ and $t(a)$ are called its start vertex and end vertex, respectively, and a is called an arrow from $s(a)$ to $t(a)$. For any $n \geq 0$, an n -path or a path of length n in the quiver Q is an ordered sequence of arrows $p = a_n a_{n-1} \cdots a_1$ with $t(a_i) = s(a_{i+1})$ for all $1 \leq i \leq n-1$. Note that a 0-path is exactly a vertex and a 1-path is exactly an arrow. In this case, we define $s(p) = s(a_1)$, the start vertex of p , and $t(p) = t(a_n)$, the end vertex of p . For a 0-path x , we have $s(x) = t(x) = x$. Let Q_n be the set of n -paths. Let ${}^y Q_n^x$ denote the set of all n -paths from x to y , $x, y \in Q_0$. That is, ${}^y Q_n^x = \{p \in Q_n \mid s(p) = x, t(p) = y\}$.

A quiver Q is *finite* if Q_0 and Q_1 are finite sets. A quiver Q is *locally finite* if ${}^y Q_1^x$ is a finite set for any $x, y \in Q_0$.

Let G be a group. Let $\mathcal{K}(G)$ denote the set of conjugate classes in G . A formal sum $r = \sum_{C \in \mathcal{K}(G)} r_C C$ of conjugate classes of G with cardinal number coefficients is called a *ramification* (or *ramification data*) of G , i.e. for any $C \in \mathcal{K}(G)$, r_C is a cardinal number. In particular, a formal sum $r = \sum_{C \in \mathcal{K}(G)} r_C C$ of conjugate classes of G with non-negative integer coefficients is a ramification of G .

For any ramification r and a $C \in \mathcal{K}(G)$, since r_C is a cardinal number, we can choose a set $I_C(r)$ such that its cardinal number is r_C without loss generality. Let $\mathcal{K}_r(G) := \{C \in \mathcal{K}(G) \mid r_C \neq 0\} = \{C \in \mathcal{K}(G) \mid I_C(r) \neq \emptyset\}$. If there exists a ramification r of G such that the cardinal number of ${}^y Q_1^x$ is equal to r_C for any $x, y \in G$ with $x^{-1}y \in C \in \mathcal{K}(G)$, then Q is called a *Hopf quiver with respect to the ramification data r* . In this case, there is a bijection from $I_C(r)$ to ${}^y Q_1^x$, and hence we write ${}^y Q_1^x = \{a_{y,x}^{(i)} \mid i \in I_C(r)\}$ for any $x, y \in G$ with $x^{-1}y \in C \in \mathcal{K}(G)$. Denote by (Q, G, r) the Hopf quiver of G with respect to r .

If $\phi : A \rightarrow A'$ is an algebra homomorphism and (M, α^-) is a left A' -module, then

M becomes a left A -module with the A -action given by $a \cdot x = \phi(a) \cdot x$ for any $a \in A$, $x \in M$, called a pullback A -module through ϕ , written as ${}_{\phi}M$. Dually, if $\phi : C \rightarrow C'$ is a coalgebra homomorphism and (M, δ^-) is a left C -comodule, then M is a left C' -comodule with the C' -comodule structure given by $\delta'^- := (\phi \otimes \text{id})\delta^-$, called a push-out C' -comodule through ϕ , written as ${}^{\phi}M$.

If B is a Hopf algebra and M is a B -Hopf bimodule, then we say that (B, M) is a Hopf bimodule. For any two Hopf bimodules (B, M) and (B', M') , if ϕ is a Hopf algebra homomorphism from B to B' and ψ is simultaneously a B -bimodule homomorphism from M to ${}_{\phi}M'_{\phi}$ and a B' -bicomodule homomorphism from ${}^{\phi}M^{\phi}$ to M' , then (ϕ, ψ) is called a pull-push Hopf bimodule homomorphism. If ψ is a bijection, then we say that (ϕ, ψ) is a pull-push Hopf bimodule isomorphism, written as $(B, M) \cong (B', M')$ as pull-push Hopf bimodules. In particular, if $B = B'$ we also write $M \cong M'$ as pull-push B -Hopf bimodules, in short. Similarly, we say that (B, M) and (B, X) are a Yetter-Drinfeld module and a Yetter-Drinfeld Hopf algebra if M is a Yetter-Drinfeld B -module and X is a braided Hopf algebra in Yetter-Drinfeld category ${}^B_B\mathcal{YD}$, respectively. For any two Yetter-Drinfeld modules (B, M) and (B', M') , if ϕ is a Hopf algebra homomorphism from B to B' , and ψ is simultaneously a left B -module homomorphism from M to ${}_{\phi}M'$ and a left B' -comodule homomorphism from ${}^{\phi}M$ to M' , then (ϕ, ψ) is called a pull-push Yetter-Drinfeld module homomorphism. For any two Yetter-Drinfeld Hopf algebra (B, X) and (B', X') , if ϕ is a Hopf algebra homomorphism from B to B' , ψ is simultaneously a left B -module homomorphism from X to ${}_{\phi}X'$ and a left B' -comodule homomorphism from ${}^{\phi}X$ to X' , meantime, ψ also is algebra and coalgebra homomorphism from X to X' , then (ϕ, ψ) is called a pull-push Yetter-Drinfeld Hopf algebra homomorphism (see [ZZC, the remark after Th.4]).

Let A be an algebra and M be an A -bimodule. Then the tensor algebra $T_A(M)$ of M over A is a graded algebra with $T_A(M)_{(0)} = A$, $T_A(M)_{(1)} = M$ and $T_A(M)_{(n)} = \otimes_A^n M$ for $n > 1$. That is, $T_A(M) = A \oplus (\bigoplus_{n>0} \otimes_A^n M)$ (see [N]). Let D be another algebra. If h is an algebra map from A to D and f is an A -bimodule map from M to $D = {}_h D_h$, then by the universal property of $T_A(M)$ (see [N, Proposition 1.4.1]) there is a unique algebra map $T_A(h, f) : T_A(M) \rightarrow D$ such that $T_A(h, f)\iota_0 = h$ and $T_A(h, f)\iota_1 = f$. One can easily see that $T_A(h, f) = h + \sum_{n>0} \mu^{n-1} T_n(f)$. For the details, the reader is directed to [N, Section 1.4] or [ZZC]. Dually, let C be a coalgebra and let M be a C -bicomodule. Then the cotensor coalgebra $T_C^c(M)$ of M over C is a graded coalgebra with $T_C^c(M)_{(0)} = C$, $T_C^c(M)_{(1)} = M$ and $T_C^c(M)_{(n)} = \square_C^n M$ for $n > 1$. That is, $T_C^c(M) = C \oplus (\bigoplus_{n>0} \square_C^n M)$ (see [N] or [ZZC]). If B is a Hopf algebra and M is a B -Hopf bimodule, then both $T_B(M)$ and $T_B^c(M)$ are graded Hopf algebras. Furthermore, subalgebra generated by H and M in $T_B^c(M)$, written as $H[M]$, is a Hopf subalgebra of $T_B^c(M)$ and $H[M]$ is called a Hopf

algebra of type one.

1 Classification of Quiver Hopf Algebras

Definition 1.1. $(G, r, \vec{\rho}, u)$ is called a ramification system with irreducible representations (or RSR in short), if r is a ramification of G ; u is a map from $\mathcal{K}(G)$ to G with $u(C) \in C$ for any $C \in \mathcal{K}(G)$; $I_C(r, u)$ and $J_C(i)$ are sets with $|J_C(i)| = \deg(\rho_C^{(i)})$ and $I_C(r) = \{(i, j) \mid i \in I_C(r, u), j \in J_C(i)\}$ for any $C \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$; $\vec{\rho} = \{\rho_C^{(i)}\}_{i \in I_C(r, u), C \in \mathcal{K}_r(G)} \in \prod_{C \in \mathcal{K}_r(G)} (\widehat{Z_{u(C)}})^{|I_C(r, u)|}$ with $\rho_C^{(i)} \in \widehat{Z_{u(C)}}$ for any $i \in I_C(r, u), C \in \mathcal{K}_r(G)$.

$\text{RSR}(G, r, \vec{\rho}, u)$ and $\text{RSR}(G', r', \vec{\rho}', u')$ are said to be isomorphic if the following conditions are satisfied:

- There exists a group isomorphism $\phi : G \rightarrow G'$.
- For any $C \in \mathcal{K}(G)$, there exists an element $h_C \in G$ such that $\phi(h_C^{-1}u(C)h_C) = u'(\phi(C))$.
- For any $C \in \mathcal{K}_r(G)$, there exists a bijective map $\phi_C : I_C(r, u) \rightarrow I_{\phi(C)}(r', u')$ such that $\rho_C^{(i)} \cong \rho'_{\phi(C)}^{(\phi_C(i))} \phi_{h_C}$ as representations of $kZ_{u(C)}$ for all $i \in I_C(r, u)$, where $\phi_{h_C}(h) = \phi(h_C^{-1}hh_C)$ for any $h \in G$.

Remark. Assume that $G = G'$, $r = r'$, $u(C) = u'(C)$ and $I_C(r, u) = I_C(r', u')$ for any $C \in \mathcal{K}_r(G)$. If there is a permutation ϕ_C on $I_C(r, u)$ for any $C \in \mathcal{K}_r(G)$ such that $\rho'_{\phi(C)}^{(\phi_C(i))} \cong \rho_C^{(i)}$ for all $i \in I_C(r, u)$, then obviously $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G, r, \vec{\rho}', u)$.

Example 1.2. Assume that k is a complex field and $G = \mathbb{S}_3$, then there are 3 elements in $\mathcal{K}(G)$, which are $\{(1)\}$, $\{(12), (13), (23)\}$, $\{(123), (132)\}$, and there are 3 non-isomorphic irreducible representations, which are two 1 dimensional irreducible representations ϵ and sgn ; one 2 dimensional irreducible representation ρ . Obviously $Z_{u(\{1\})} = G$. The set $\{\text{RSR}(G, r, \vec{\rho}, u) \mid \vec{\rho} = \rho, (\text{sgn}, \text{sgn}), (\epsilon, \text{sgn}), (\text{sgn}, \epsilon), (\epsilon, \epsilon)\}$ is all RSR's with $r = r_C r$ and $C = \{(1)\}$. The set $\{\text{RSR}(G, r, \vec{\rho}, u) \mid \vec{\rho} = \rho, (\text{sgn}, \text{sgn}), (\epsilon, \text{sgn}), (\epsilon, \epsilon)\}$ is all representatives of isomorphic classes of RSR's with $r = r_C r$ and $C = \{(1)\}$. Furthermore, when $\vec{\rho} = \rho$, we can set $I_C(r, u) = \{1\}$ and $J_C(1) = \{1\}$. In this case $\rho_C^{(1)} = \rho$. When $\vec{\rho} = (\epsilon, \text{sgn})$, we can set $I_C(r, u) = \{1, 2\}$ and $J_C(1) = J_C(2) = \{1\}$. In this case $\rho_C^{(1)} = \epsilon$, $\rho_C^{(2)} = \text{sgn}$.

Let

$$G = \bigcup_{\theta \in \Theta_C} Z_{u(C)} g \theta, \quad (1.1)$$

where Θ_C is an index set, be a coset decomposition of $Z_{u(C)}$ in G . It is easy to check that $|\Theta_C| = |C|$. We always assume that the representative element of the coset $Z_{u(C)}$ is the

identity 1 of G . For any $h \in G$ and $\theta \in \Theta_C$, there exist unique $h' \in Z_{u(C)}$ and $\theta' \in \Theta_C$ such that $g_\theta h = h' g_{\theta'}$. Let $\zeta_\theta(h) = h'$. Then we have

$$g_\theta h = \zeta_\theta(h) g_{\theta'}. \quad (1.2)$$

Proposition 1.3. *If N is a kG -Hopf bimodule, then there exist a Hopf quiver (Q, G, r) , an $\text{RSR}(G, r, \overrightarrow{\rho}, u)$ and a kG -Hopf bimodule $(kQ_1^c, \alpha^-, \alpha^+)$ with*

$$\alpha^-(h \otimes a_{y,x}^{(i,j)}) := h \cdot a_{y,x}^{(i,j)} = a_{hy,hx}^{(i,j)}, \quad \alpha^+(a_{y,x}^{(i,j)} \otimes h) := a_{y,x}^{(i,j)} \cdot h = \sum_{s \in J_C(i)} k_{C,h}^{(i,j,s)} a_{yh,xh}^{(i,s)} \quad (1.3)$$

for some $k_{C,h}^{(i,j,s)} \in k$ such that $N \cong (kQ_1^c, \alpha^-, \alpha^+)$ as kG -Hopf bimodules, where $x, y, h \in G$ with $x^{-1}y = g_\theta^{-1}u(C)g_\theta$, ζ_θ is given by [ZZC, (0.3)], $C \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, $j \in J_C(i)$, $x_C^{(i,j)} \cdot \zeta_\theta(h) = \sum_{s \in J_C(i)} k_{C,h}^{(i,j,s)} x_C^{(i,s)}$.

Proof. Since N is a kG -Hopf bimodule, there exists an object $\prod_{C \in \mathcal{K}(G)} M(C)$ in $\prod_{C \in \mathcal{K}(G)} \mathcal{M}_{kZ_{u(C)}}$ such that $M(C)$ is a $kZ_{u(C)}$ -module for any $C \in \mathcal{K}(G)$ and $N \cong \bigoplus_{y=xg_\theta^{-1}u(C)g_\theta, x,y \in G} x \otimes M(C) \otimes_{kZ_{u(C)}} g_\theta$ as kG -Hopf bimodules by [CR97] or [ZZC, Th. 1]. Let $r = \sum_{C \in \mathcal{K}(G)} r_C C$ with $r_C = \dim M(C)$ for any $C \in \mathcal{K}(G)$. Notice that $\dim M(C)$ denotes the cardinal number of a basis of $M(C)$ when $M(C)$ is infinite dimensional. Since $M(C)$ is a $kZ_{u(C)}$ -module and $kZ_{u(C)}$ is semisimple, there exists a family of irreducible representations $\{(X_C^{(i)}, \rho_C^{(i)}) \mid i \in I_C(r, u)\}$ such that $M(C) = \bigoplus_{i \in I_C(r, u)} (X_C^{(i)}, \rho_C^{(i)})$. Let $\{x_C^{(i,j)} \mid j \in J_C(i)\}$ be a k -basis of $X_C^{(i)}$ for any $i \in I_C(r, u)$. Then for any $h \in G$ there are some $k_{C,h}^{(i,j,s)} \in k$ such that $x_C^{(i,j)} \cdot \zeta_\theta(h) = \sum_{s \in J_C(i)} k_{C,h}^{(i,j,s)} x_C^{(i,s)}$ for all $i \in I_C(r, u)$ and $j \in J_C(i)$ since $x_C^{(i,j)} \cdot \zeta_\theta(h) \in X_C^{(i)}$.

It remains to show that $(kQ_1^c, \alpha^-, \alpha^+)$ is isomorphic to $\bigoplus_{y=xg_\theta^{-1}u(C)g_\theta, x,y \in G} x \otimes M(C) \otimes_{kZ_{u(C)}} g_\theta$ as kG -Hopf bimodules. Observe that there is a canonical kG -bicomodule isomorphism $\varphi : kQ_1 \rightarrow \bigoplus_{y=xg_\theta^{-1}u(C)g_\theta, x,y \in G} x \otimes M(C) \otimes_{kZ_{u(C)}} g_\theta$ given by

$$\varphi(a_{y,x}^{(i,j)}) = x \otimes x_C^{(i,j)} \otimes_{kZ_{u(C)}} g_\theta \quad (1.4)$$

where $x, y \in G$ with $x^{-1}y = g_\theta^{-1}u(C)g_\theta$, $C \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$ and $j \in J_C(r)$. See

$$\begin{aligned} \varphi(\alpha^-(h \otimes a_{y,x}^{(i,j)})) &= \varphi(a_{hy,hx}^{(i,j)}) = hx \otimes x_C^{(i,j)} \otimes_{kZ_{u(C)}} g_\theta \\ &= h \cdot (x \otimes x_C^{(i,j)} \otimes_{kZ_{u(C)}} g_\theta) \quad (\text{see [ZZC, (1.2)]}) \\ &= h \cdot \varphi(a_{y,x}^{(i,j)}) \quad . \end{aligned}$$

Thus φ is a left kG -module isomorphism. See

$$\begin{aligned}
\alpha^+(\varphi(a_{y,x}^{(i,j)}) \otimes h) &= xh \otimes x_C^{(i,j)} \cdot \zeta_\theta(h) \otimes g_{\theta'} \\
&= xh \otimes \left(\sum_{s \in J_C(i)} k_{C,h}^{(i,j,s)} x_C^{(i,s)} \right) \otimes g_{\theta'} \\
&= \varphi \left(\sum_{s \in J_C(i)} k_{C,h}^{(i,j,s)} a_{yh,xh}^{(i,s)} \right) \\
&= \varphi(\alpha^+(a_{y,x}^{(i,j)} \otimes h)) \quad (\text{by (1.3)}).
\end{aligned}$$

Consequently, φ is a kG -Hopf bimodule isomorphism. \square

Let $(kQ_1^c, G, r, \vec{\rho}, u)$ denote the kG -Hopf bimodule $(kQ_1^c, \alpha^-, \alpha^+)$ given in Proposition 1.3. Furthermore, if (kQ_1^c, kQ_1^a) is an arrow dual pairing, i.e. kQ_1^c is isomorphic to the dual of kQ_1^a as kG -Hopf bimodules or kQ_1^a is isomorphic to the dual of kQ_1^c as $(kG)^*$ -Hopf bimodules under the isomorphisms in [ZZC, Lemma 1.7] (cf [ZZC, the argument before Def. 1.8]), then we denote the $(kG)^*$ -Hopf bimodule kQ_1^a by $(kQ_1^a, G, r, \vec{\rho}, u)$. We obtain six quiver Hopf algebras $kQ^c(G, r, \vec{\rho}, u)$, $kQ^s(G, r, \vec{\rho}, u)$, $kG[kQ_1^c, G, r, \vec{\rho}, u]$, $kQ^a(G, r, \vec{\rho}, u)$, $kQ^{sc}(G, r, \vec{\rho}, u)$, $(kG)^*[kQ_1^a, G, r, \vec{\rho}, u]$, called the quiver Hopf algebras determined by RSR($G, r, \vec{\rho}, u$).

From Proposition 1.3, it seems that the right kG -action on $(kQ_1^c, G, r, \vec{\rho}, u)$ depends on the choice of the set $\{g_\theta \mid \theta \in \Theta_C\}$ of coset representatives of $Z_{u(C)}$ in G (see, (1.1) or [ZZC, (0.1)]). The following lemma shows that $(kQ_1^c, G, r, \vec{\rho}, u)$ is, in fact, independent of the choice of the coset representative set $\{g_\theta \mid \theta \in \Theta_C\}$, up to kG -Hopf bimodule isomorphisms. For a while, we write $(kQ_1^c, G, r, \vec{\rho}, u) = (kQ_1^c, G, r, \vec{\rho}, u, \{g_\theta\})$ given before. Now let $\{h_\theta \in G \mid \theta \in \Theta_C\}$ be another coset representative set of $Z_{u(C)}$ in G for any $C \in \mathcal{K}(G)$. That is,

$$G = \bigcup_{\theta \in \Theta_C} Z_{u(C)} h_\theta. \quad (1.5)$$

Lemma 1.4. *With the above notations, $(kQ_1^c, G, r, \vec{\rho}, u, \{g_\theta\})$ and $(kQ_1^c, G, r, \vec{\rho}, u, \{h_\theta\})$ are isomorphic kG -Hopf bimodules.*

Proof. We may assume $Z_{u(C)} h_\theta = Z_{u(C)} g_\theta$ for any $C \in \mathcal{K}(G)$ and $\theta \in \Theta_C$. Then $g_\theta h_\theta^{-1} \in Z_{u(C)}$. Now let $x, y, h \in G$ with $x^{-1}y = g_\theta^{-1}u(C)g_\theta$. Then $x^{-1}y = h_\theta^{-1}(g_\theta h_\theta^{-1})^{-1}u(C)(g_\theta h_\theta^{-1})h_\theta = h_\theta^{-1}u(C)h_\theta$ and

$$h_\theta h = (h_\theta g_\theta^{-1})g_\theta h = (h_\theta g_\theta^{-1})\zeta_\theta(h)g_{\theta'} = (h_\theta g_\theta^{-1})\zeta_\theta(h)(g_{\theta'} h_{\theta'}^{-1})h_{\theta'}, \quad (1.6)$$

where $g_\theta h = \zeta_\theta(h)g_{\theta'}$.

For any $C \in \mathcal{K}(G)$, let $M(C)$ be a right $kZ_{u(C)}$ -module. Let $N := \bigoplus_{y=xg_\theta^{-1}u(C)g_\theta, x,y \in G} x \otimes M(C) \otimes_{kZ_{u(C)}} g_\theta$ and $M := \bigoplus_{y=xh_\theta^{-1}u(C)h_\theta, x,y \in G} x \otimes M(C) \otimes_{kZ_{u(C)}} h_\theta$ be two kG -Hopf bimodules. It is sufficient to show $N \cong M$ as kG -Hopf bimodules by the proof of Proposition 1.3.

Considering $x \otimes w \otimes_{kZ_{u(C)}} g_\theta = x \otimes w \cdot g_\theta h_\theta^{-1} \otimes_{kZ_{u(C)}} h_\theta$, we have that $f : N \rightarrow M$ given by

$$f(x \otimes w \otimes_{kZ_{u(C)}} g_\theta) = x \otimes w \cdot g_\theta h_\theta^{-1} \otimes_{kZ_{u(C)}} h_\theta$$

for any $w \in M(C)$, any $x, y \in G$ with $x^{-1}y = g_\theta^{-1}u(C)g_\theta$, $C \in \mathcal{K}_r(G)$ and $i \in I_C(r, u)$, is a k -linear isomorphism. It is clear that f is a kG -bicomodule isomorphism and a left kG -module isomorphism from N to M . See

$$\begin{aligned} & (f(x \otimes x_C^{(i,j)} \otimes_{kZ_{u(C)}} g_\theta)) \cdot h \\ &= (x \otimes (x_C^{(i,j)}) \rho_C^{(i)}(g_\theta h_\theta^{-1}) \otimes_{kZ_{u(C)}} h_\theta) \cdot h \\ &= xh \otimes (x_C^{(i,j)}) \rho_C^{(i)}(\zeta_\theta(h)) \rho_C^{(i)}(g_{\theta'} h_{\theta'}^{-1}) \otimes_{kZ_{u(C)}} h_{\theta'} \quad (\text{by (1.6)}) \\ &= f((x \otimes x_C^{(i,j)} \otimes_{kZ_{u(C)}} g_\theta) \cdot h), \end{aligned}$$

for any $x, y, h \in G$, $i \in I_C(r, u)$, $j \in J_C(i)$, $C \in \mathcal{K}_r(G)$ with $x^{-1}y = g_\theta^{-1}u(C)g_\theta$. Thus f is a right kG -module homomorphism. \square

Now we state one of our main result, which classifies quiver Hopf algebras.

Theorem 2. *Let $(G, r, \vec{\rho}, u)$ and $(G', r', \vec{\rho}', u')$ be two RSR's. Then the following statements are equivalent:*

- (i) $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G', r', \vec{\rho}', u')$.
- (ii) *There exists a Hopf algebra isomorphism $\phi : kG \rightarrow kG'$ such that $(kQ_1^c, G, r, \vec{\rho}, u) \cong \phi^{-1}((kQ_1'^c, G', r', \vec{\rho}', u'))_\phi^{\phi^{-1}}$ as kG -Hopf bimodules.*
- (iii) $kQ^c(G, r, \vec{\rho}, u) \cong kQ'^c(G', r', \vec{\rho}', u')$.
- (iv) $kQ^s(G, r, \vec{\rho}, u) \cong kQ'^s(G', r', \vec{\rho}', u')$.
- (v) $kG[kQ_1^c, G, r, \vec{\rho}, u] \cong kG'[kQ_1'^c, G', r', \vec{\rho}', u']$.

Furthermore, if Q is finite, then the above are equivalent to the following:

- (vi) $kQ^a(G, r, \vec{\rho}, u) \cong kQ'^a(G', r', \vec{\rho}', u')$.
- (vii) $kQ^{sc}(G, r, \vec{\rho}, u) \cong kQ'^{sc}(G', r', \vec{\rho}', u')$.
- (viii) $(kG)^*[kQ_1^a, G, r, \vec{\rho}, u] \cong (kG')^*[kQ_1'^a, G', r', \vec{\rho}', u']$. Notice that the isomorphisms above are ones of graded Hopf algebras but (i) (ii).

Proof. By [ZZC, Lemma 1.5, Lemma 1.6], we only have to prove (i) \Leftrightarrow (ii).

(i) \Rightarrow (ii). Assume that $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G', r', \vec{\rho}', u')$. Let $(X_C^{(i)}, \rho_C^{(i)})$ and $(X_{C'}^{(i')}, \rho_{C'}^{(i')})$ be irreducible representations over $Z_{u(C)}$ and $Z_{u'(C')}$, respectively. Then there exist a group isomorphism $\phi : G \rightarrow G'$, an element $h_C \in G$ such that $\phi(h_C^{-1}u(C)h_C) = u'(\phi(C))$ for any $C \in \mathcal{K}(G)$ and a bijective map $\phi_C : I_C(r, u) \rightarrow I_{\phi(C)}(r', u')$ such that

$$(X_C^{(i)}, \rho_C^{(i)}) \stackrel{\xi_C^{(i)}}{\cong} (X_{\phi(C)}^{(\phi_C(i))}, \rho_{\phi(C)}^{(\phi_C(i))} \phi_{h_C}) \text{ as right } kZ_{u(C)}\text{-modules for all } i \in I_C(r, u).$$

Now let $G = \bigcup_{\theta \in \Theta_C} Z_{u(C)}g_\theta$ as in (1.1) or [ZZC, (0.1)] for any $C \in \mathcal{K}(G)$. Let $x, y, h \in G$ with $x^{-1}y = g_\theta^{-1}u(C)g_\theta$, $C \in \mathcal{K}_r(G)$ and $\theta \in \Theta_C$. Assume that $g_\theta h_C^{-1} = \zeta_\theta(h_C^{-1})g_\eta$,

$g_\theta h = \zeta_\theta(h)g_{\theta'}$, $g_\eta(h_C h h_C^{-1}) = \zeta_\eta(h_C h h_C^{-1})g_{\eta'}$ and $g_{\theta'} h_C^{-1} = \zeta_{\theta'}(h_C^{-1})g_{\theta''}$ with $\zeta_\theta(h_C^{-1})$, $\zeta_\theta(h)$, $\zeta_\eta(h_C h h_C^{-1})$, $\zeta_{\theta'}(h_C^{-1}) \in Z_{u(C)}$ and $\eta, \theta', \eta', \theta'' \in \Theta_C$. Then we have

$$g_\theta h h_C^{-1} = \zeta_\theta(h)g_{\theta'} h_C^{-1} = \zeta_\theta(h)\zeta_{\theta'}(h_C^{-1})g_{\theta''} \quad (1.7)$$

and

$$g_\theta h h_C^{-1} = (g_\theta h_C^{-1})(h_C h h_C^{-1}) = \zeta_\theta(h_C^{-1})g_\eta(h_C h h_C^{-1}) = \zeta_\theta(h_C^{-1})\zeta_\eta(h_C h h_C^{-1})g_{\eta'}. \quad (1.8)$$

It follows that

$$\theta'' = \eta' \quad \text{and} \quad \zeta_\theta(h)\zeta_{\theta'}(h_C^{-1}) = \zeta_\theta(h_C^{-1})\zeta_\eta(h_C h h_C^{-1}). \quad (1.9)$$

Moreover, we have $(xh)^{-1}(yh) = h^{-1}g_\theta^{-1}u(C)g_\theta h = g_{\theta'}^{-1}u(C)g_{\theta'}$ and $g_\theta = g_\theta h_C^{-1}h_C = \zeta_\theta(h_C^{-1})g_\eta h_C$. Thus

$$\begin{aligned} \phi(x)^{-1}\phi(y) &= \phi(x^{-1}y) = \phi(g_\theta^{-1}u(C)g_\theta) = \phi(h_C^{-1}g_\eta^{-1}u(C)g_\eta h_C) \\ &= \phi(h_C^{-1}g_\eta^{-1}h_C)\phi(h_C^{-1}u(C)h_C)\phi(h_C^{-1}g_\eta h_C) \\ &= \phi(h_C^{-1}g_\eta h_C)^{-1}u'(\phi(C))\phi(h_C^{-1}g_\eta h_C). \end{aligned}$$

We also have

$$\begin{aligned} \phi(h_C^{-1}g_\eta h_C)\phi(h) &= \phi(h_C^{-1}g_\eta h_C h h_C^{-1}h_C) \\ &= \phi(h_C^{-1}\zeta_\eta(h_C h h_C^{-1})g_\eta h_C) \\ &= \phi(h_C^{-1}\zeta_\eta(h_C h h_C^{-1})h_C)\phi(h_C^{-1}g_\eta h_C). \end{aligned} \quad (1.10)$$

Since

$$\begin{aligned} \phi_{h_C}(g_{\eta'})\phi(xh)^{-1}\phi(yh) &= \phi(h_C^{-1}(\zeta_\eta(h_C h h_C^{-1}))^{-1}(\zeta_\theta(h_C^{-1}))^{-1}u(C)g_\theta h) \quad (\text{by 1.8}) \\ &= u'(\phi(C))\phi_{h_C}(g_{\eta'}) \quad (\text{by 1.8}), \end{aligned}$$

$$\phi(xh)^{-1}\phi(yh) = \phi_{h_C}(g_{\eta'})^{-1}u'(\phi(C))\phi_{h_C}(g_{\eta'}). \quad (1.11)$$

It is clear

$$G' = \bigcup_{\theta \in \Theta_C} Z_{u'(\phi(C))}(\phi(h_C^{-1}g_\theta h_C)) \quad (1.12)$$

is a coset decomposition of $Z_{u'(\phi(C))}$ in G' for any $\phi(C) \in \mathcal{K}(G')$.

Let

$$\begin{aligned} N &:= \bigoplus_{y=xg_\theta^{-1}u(C)g_\theta, \ x, y \in G} x \otimes N(C) \otimes_{kZ_{u(C)}} g_\theta \quad \text{and} \\ M &:= \bigoplus_{\phi(y)=\phi(x)\phi_{h_C}(g_\eta^{-1})u'(\phi(C))\phi_{h_C}(g_\eta), \ x, y \in G} \phi(x) \otimes M(\phi(C)) \otimes_{kZ_{u'(\phi(C))}} \phi_{h_C}(g_\eta) \end{aligned}$$

with $N(C) := \bigoplus_{i \in I_C(r,u)} X_C^{(i)}$ and $M(\phi(C)) := \bigoplus_{i \in I_C(r,u)} X_{\phi(C)}'^{(\phi_C(i))}$. It suffices to show $N \cong {}^{\phi^{-1}}M^{\phi^{-1}}$ as kG -Hopf bimodules by the proof of Proposition 1.3.

Considering $x \otimes (w)\rho_C^{(i)}((\zeta_\theta(h_C^{-1}))^{-1}) \otimes_{kZ_{u(C)}} g_\theta = x \otimes w \otimes_{kZ_{u(C)}} g_\eta h_c$, we have that $\psi : N \rightarrow M$ given by

$$\psi(x \otimes (w)\rho_C^{(i)}((\zeta_\theta(h_C^{-1}))^{-1}) \otimes_{kZ_{u(C)}} g_\theta) = \phi(x) \otimes \xi_C^{(i)}(w) \otimes_{kZ_{u'(\phi(C))}} \phi_{h_C}(g_\eta)$$

for any $x, y \in G$ with $x^{-1}y = g_\theta^{-1}u(C)g_\theta$, and $i \in I_C(r, u)$, $w \in X_C^{(i)}$, where $C \in \mathcal{K}_r(G)$ and $g_\theta h_C^{-1} = \zeta_\theta(h_C^{-1})g_\eta$ with $\zeta_\theta(h_C^{-1}) \in Z_{u(C)}$ and $\theta, \eta \in \Theta_C$, is a k -linear isomorphism. It is clear that ψ is a homomorphism not only of kG -bicomodules from N to ${}^{\phi^{-1}}M^{\phi^{-1}}$ but also of left kG -modules from N to ${}_\phi M$.

For any $h \in G$ and $w \in X_C^{(i)}$, see

$$\begin{aligned} & \psi((x \otimes (w)\rho_C^{(i)}((\zeta_\theta(h_C^{-1}))^{-1}) \otimes_{kZ_{u(C)}} g_\theta) \cdot h) \\ &= \psi((xh \otimes (w)\rho_C^{(i)}((\zeta_\theta(h_C^{-1}))^{-1}\zeta_\theta(h)) \otimes_{kZ_{u(C)}} g_{\theta'}) \\ &= \phi(xh) \otimes \xi_C^{(i)}((w)\rho_C^{(i)}((\zeta_\theta(h_C^{-1}))^{-1}\zeta_\theta(h)\zeta_{\theta'}(h_C^{-1}))) \otimes_{kZ_{u'(\phi(C))}} \phi_{h_C}(g_{\eta'}) \quad (\text{by (1.11)}) \\ &= \phi(x)\phi(h) \otimes (\xi_C^{(i)}(w))\rho_{\phi(C)}^{(\phi_C(i))}\phi_{h_C}((\zeta_\theta(h_C^{-1}))^{-1}\zeta_\theta(h)\zeta_{\theta'}(h_C^{-1}))) \otimes_{kZ_{u'(\phi(C))}} \phi_{h_C}(g_{\eta'}) \\ & \quad (\text{by Def. 1.1}) \end{aligned}$$

and

$$\begin{aligned} & \psi(x \otimes (w)\rho_C^{(i)}((\zeta_\theta(h_C^{-1}))^{-1}) \otimes_{kZ_{u(C)}} g_\theta) \cdot \phi(h) \\ &= (\phi(x) \otimes \xi_C^{(i)}(w) \otimes_{kZ_{u'(\phi(C))}} \phi_{h_C}(g_\eta)) \cdot \phi(h) \\ &= \phi(x)\phi(h) \otimes (\xi_C^{(i)}(w))\rho_{\phi(C)}^{(\phi_C(i))}\phi_{h_C}(\zeta_\eta(h_C h h_C^{-1})) \otimes_{kZ_{u'(\phi(C))}} \phi_{h_C}(g_{\eta'}) \quad (\text{by (1.10)}) \\ &= \phi(x)\phi(h) \otimes (\xi_C^{(i)}(w))\rho_{\phi(C)}^{(\phi_C(i))}\phi_{h_C}((\zeta_\theta(h_C^{-1}))^{-1}\zeta_\theta(h)\zeta_{\theta'}(h_C^{-1}))) \otimes_{kZ_{u'(\phi(C))}} \phi_{h_C}(g_{\eta'}) \\ & \quad (\text{by (1.9)}), \end{aligned}$$

which show that ψ is a right kG -module homomorphism.

(ii) \Rightarrow (i). Assume that there exist a Hopf algebra isomorphism $\phi : kG \rightarrow kG'$ and a kG -Hopf bimodule isomorphism $\psi : (kQ_1^c, G, r, \vec{\rho}, u) \rightarrow {}^{\phi^{-1}}(kQ_1'^c, G', r', \vec{\rho}', u')^{\phi^{-1}}$. Then $\phi : G \rightarrow G'$ is a group isomorphism. Let $C \in \mathcal{K}(G)$. Then $\phi(u(C)), u'(\phi(C)) \in \phi(C) \in \mathcal{K}(G')$, and hence $u'(\phi(C)) = \phi(h_C)^{-1}\phi(u(C))\phi(h_C) = \phi(h_C^{-1}u(C)h_C)$ for some $h_C \in G$. Since ψ is a kG' -bicomodule isomorphism from ${}^\phi(kQ_1^c, G, r, \vec{\rho}, u)^\phi$ to $(kQ_1'^c, G', r', \vec{\rho}', u')$ and $\phi(h_C^{-1}u(C)h_C) = u'(\phi(C))$, by restriction one gets a k -linear isomorphism

$$\psi_C : {}^{h_C^{-1}u(C)h_C}(kQ_1)^1 \rightarrow {}^{u'(\phi(C))}(kQ_1')^1, \quad x \mapsto \psi(x).$$

We also have a k -linear isomorphism

$$f_C : {}^{u(C)}(kQ_1)^1 \rightarrow {}^{h_C^{-1}u(C)h_C}(kQ_1)^1, \quad x \mapsto h_C^{-1} \cdot x \cdot h_C.$$

Since $\phi(h_C^{-1}u(C)h_C) = u'(\phi(C))$ and $h_C^{-1}Z_{u(C)}h_C = Z_{h_C^{-1}u(C)h_C}$, one gets $\phi(h_C^{-1}Z_{u(C)}h_C) = Z_{u'(\phi(C))}$. Hence ϕ_{h_C} is an algebra isomorphism from $kZ_{u(C)}$ to $kZ_{u'(\phi(C))}$ by sending h to $\phi(h_C^{-1}hh_C)$. Using the hypothesis that ψ is a kG -bimodules homomorphism from $(kQ_1^c, G, r, \vec{\rho}, u)$ to ${}_{\phi}(kQ_1'^c, G', r', \vec{\rho}', u')_{\phi}$, one can easily check that the composition $\psi_C f_C$ is a right $kZ_{u(C)}$ -module isomorphism from $({}^{u(C)}(kQ_1)^1, \triangleleft)$ to $(({}^{u'(\phi(C))}(kQ_1')^1)_{\phi_{h_C}}, \triangleleft)$. Indeed, For any $z \in {}^{u(C)}(kQ_1)^1$, see

$$\begin{aligned} \psi_C f_C(z) \triangleleft \phi_{h_C}(h) &= \psi_C(z \triangleleft h_C) \triangleleft \phi_{h_C}(h) \\ &= \psi_C(z \triangleleft h_C \phi_{h_C}(h)) \quad (\text{since } \psi \text{ is a bimodule homomorphism}) \\ &= \psi_C f_C(z \triangleleft h). \end{aligned}$$

Obviously, both ${}^{u(C)}(kQ_1)^1$ and $({}^{u'(\phi(C))}(kQ_1')^1)_{\phi_{h_C}}$ are semisimple right $kZ_{u(C)}$ -modules. Assume ${}^{u(C)}(kQ_1)^1 \cong \bigoplus_{i \in I_C(r, u)} (X_C^{(i)}, \rho_C^{(i)})$ as right $kZ_{u(C)}$ -modules and ${}^{u'(\phi(C))}(kQ_1')^1 \cong \bigoplus_{j \in I_{\phi(C)}(r', u')} (X'_{\phi(C)}^{(j)}, \rho'_{\phi(C)}^{(j)})$ as right $kZ_{u'(\phi(C))}$ -modules, where $(X_C^{(i)}, \rho_C^{(i)})$ is an irreducible right $kZ_{u(C)}$ -module for any $i \in I_C(r, u)$ and $(X'_{\phi(C)}^{(j)}, \rho'_{\phi(C)}^{(j)})$ is an irreducible right $kZ_{u'(\phi(C))}$ -module for any $j \in I_{\phi(C)}(r', u')$. Therefore, there exists a bijective map $\phi_C : I_C(r, u) \rightarrow I_{\phi(C)}(r', u')$ such that $(X_C^{(i)}, \rho_C^{(i)}) \cong (X'_{\phi(C)}^{(\phi_C(i))}, \rho'_{\phi(C)}^{(\phi_C(i))} \phi_{h_C})$ as right $kZ_{u(C)}$ -modules for all $i \in I_C(r, u)$. It follows that $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G', r', \vec{\rho}', u')$. \square

Up to now we have classified the quiver Hopf algebras by means of RSR's. In other words, ramification systems with irreducible representations uniquely determine their corresponding quiver Hopf algebras up to graded Hopf algebra isomorphisms.

Proposition 1.5. *Let $\text{RSR}(G, r, \vec{\rho}, u)$ and $\text{RSR}(G, r, \vec{\rho}', u')$ be two RSR's. If $u'(C) = h_C^{-1}u(C)h_C$ and $\rho_C^{(i)} = \rho'_{\phi(C)}^{(i)} \text{ad}_{h_C}^+$ with $I_C(r, u) = I_C(r, u')$ for any $C \in \mathcal{K}_r(G)$, $i \in I_C(r, u)$, where $h_C \in G$ and $\text{ad}_{h_C}^+(g) = h_C^{-1}gh_C$, then $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G, r, \vec{\rho}', u')$.*

Proof. Let $\phi = id_G$ and $\phi_C = id_{I_C(r, u)}$ for any $C \in \mathcal{K}_r(G)$. It is clear that $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G, r, \vec{\rho}', u')$. \square

Remark: This proposition means that the choice of map u doesn't affect the classification of RSR's. That is, if we fix a map u_0 from $\mathcal{K}(G)$ to G with $u_0(C) \in C$ for any $C \in \mathcal{K}(G)$, then for any $\text{RSR}(G, r, \vec{\rho}, u)$, there exists $\text{RSR}(G, r, \vec{\rho}', u_0)$ such that $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G, r, \vec{\rho}', u_0)$.

2 The classification of pointed Hopf algebras of type one

A graded Hopf algebra $A = \bigoplus_{n=0}^{\infty} A_{(n)}$ is called to be of Nichols type, if diagram of A is a Nichols algebra over $A_{(0)}$ (the definition of diagram was in [AS98] and [ZZC, Subsection 3.1]). Furthermore, if the coradical of A is a group algebra, then A is called a pointed Hopf algebra of Nichols type.

For an $\text{RSR}(G, r, \vec{\rho}, u)$ and a kG -Hopf bimodule $(kQ_1^c, G, r, \vec{\rho}, u)$ with the module operations α^- and α^+ , define a new left kG -action on kQ_1 by

$$g \triangleright x := g \cdot x \cdot g^{-1}, \quad g \in G, x \in kQ_1,$$

where $g \cdot x = \alpha^-(g \otimes x)$ and $x \cdot g = \alpha^+(x \otimes g)$ for any $g \in G$ and $x \in kQ_1$. With this left kG -action and the original left (arrow) kG -coaction δ^- , kQ_1 is a Yetter-Drinfeld kG -module. Let $Q_1^1 := \{a \in Q_1 \mid s(a) = 1\}$, the set of all arrows with starting vertex 1. It is clear that kQ_1^1 is a Yetter-Drinfeld kG -submodule of kQ_1 , denoted by $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$.

Lemma 2.1. (i) *If H is a Hopf algebra with bijective antipode and $(B, \alpha_B^-, \delta_B^-)$ is a graded braided Hopf algebra in ${}^H_H\mathcal{YD}$ with $B_{(0)} = k1_B$, then $\text{diag}(B \# H) = B \# 1_H \cong B$ as graded braided Hopf algebras in ${}^H_H\mathcal{YD}$.*

(ii) *A is a pointed Hopf algebra of Nichols type if and only if A is isomorphic to biproduct $\mathcal{B}(V) \# kG$ with Nichols algebra $\mathcal{B}(V)$ over group algebra kG , $A_{(0)} = kG$ and $V = A_{(1)}$.*

Proof. (i) Obviously, $\text{diag}(B \# H) = B \otimes 1_H$. Define a map ψ from $\text{diag}(B \# H)$ to B by sending $x \otimes 1_H$ to x for any $x \in B$. It is easy to check that ψ is a graded braided Hopf algebra isomorphism in ${}^H_H\mathcal{YD}$.

(ii) If A is a pointed Hopf algebra of Nichols type, then $\text{diag}(A) = \mathcal{B}(V)$ is a Nichols algebra in ${}^{kG}_{kG}\mathcal{YD}$ and the coradical of A is group algebra kG . Therefore $A \cong \mathcal{B}(V) \# kG$ as graded Hopf algebras. By [AS98, Lemma 2.5], $A_{(0)} = kG$ and $A_{(1)} = V$.

Conversely, clearly, $\text{diag}(\mathcal{B}(V) \# kG) = (\mathcal{B}(V) \# kG)^{\text{co}kG} = \mathcal{B}(V) \# 1 \cong \mathcal{B}(V)$ by (i) and the coradical of $\mathcal{B}(V) \# kG$ is $(\mathcal{B}(V) \# kG)_{(0)} = kG$. \square

Lemma 2.2. *If $H = kG$ is a group algebra and M is an H -Hopf bimodule, then pointed Hopf algebra $H[M]$ of type one is a Hopf algebra of Nichols type. In particular, one-type-co-path Hopf algebra $kG[kQ_1^c, G, r, \vec{\rho}, u]$ is a pointed Hopf algebra of Nichols type and $\text{diag}(kG[kQ_1^c, G, r, \vec{\rho}, u]) = \mathcal{B}(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$.*

Proof. By Proposition 1.3 there exists an $\text{RSR}(G, r, \vec{\rho}, u)$ such that $M \cong (kQ_1^c, G, r, \vec{\rho}, u)$ as kG -Hopf bimodules. Thus $kG[M] \cong kG[kQ_1^c]$ as graded Hopf algebras by [ZZC,

Lemma 1.6]. Therefore, it is enough to show that $kG[kQ_1^c]$ is of Nichols type. Let $A := kG[kQ_1^c]$ and $R := \text{diag}(kG[kQ_1^c])$. Obviously, $R_{(0)} = k$. Now we show $R_{(1)} = kQ_1^1$. Obviously, $kQ_1^1 \subseteq R_{(1)}$. Let $\alpha = \sum_{p=1}^n k_p b^{(p)} \in R_{(1)}$, where $b^{(p)}$ is an arrow from $x^{(p)}$ to $y^{(p)}$ with $0 \neq k_p \in k$, and $b^{(1)}, b^{(2)}, \dots, b^{(n)}$ are different each other. Therefore $\sum_{p=1}^n k_p b^{(p)} \otimes 1 = \sum_{p=1}^n k_p b^{(p)} \otimes x^{(p)}$, which implies $x^{(p)} = 1$ for $1 \leq p \leq n$. Thus $\alpha \in kQ_1^1$. We next show that R is generated by $R_{(1)}$ as algebras. Let μ denote the multiplication and let B denote the algebra generated by kQ_1^1 in $kG[kQ_1^c]$. Obviously, $B \subseteq R$. It follows from argument in [ZZC, Subsection 3.1] that $\alpha^+ := \mu(id \otimes \iota_0)$ is an algebraic isomorphism from $R \# kG$ to $kG[kQ_1^c]$. For any $x, y \in G$ and any arrow $a_{y,x}$ from x to y , see

$$\begin{aligned} a_{y,x} &= x \cdot a_{x^{-1}y,1} = \mu(x \otimes a_{x^{-1}y,1}) = \mu(\alpha^+(1 \# x) \otimes \alpha^+(a_{x^{-1}y,1} \# 1)) \\ &= \alpha^+(\mu((1 \# x) \otimes (a_{x^{-1}y,1} \# 1))) = \alpha^+(x \triangleright a_{x^{-1}y,1} \# x) \in \alpha^+(B \# kG). \end{aligned}$$

Therefore $\alpha^+(B \# kG) = \alpha^+(R \# kG)$ and so $B = R$.

It is enough to show $P(R) = kQ_1^1$, where $P(R)$ denotes the set of all primitive elements in R . For any $a \in Q_1^1$ with $\delta^-(a) = y$ and $\delta^+(a) = 1$, see $\Delta_R(a) = (\omega \otimes id)\Delta_A(a) = 1 \otimes a + a \otimes 1$ (see [ZZC, Section 3] or [Ra]), i.e. $kQ_1^1 \subseteq P(R)$, where $\omega = \mu_A(id \otimes \iota_0 \pi_0 S)\Delta_A$.

Conversely, we shall show $P(R) \subseteq kQ_1^1$ by the following two steps. Obviously, $kG \cap P(R) = 0$ and $P(R)$ is a graded subspace of R .

(i) Assume that $\alpha = a_{x_n x_{n-1}} a_{x_{n-1} x_{n-2}} \cdots a_{x_1 x_0}$ is a path from vertex x_0 , via arrows $a_{x_1 x_0}, \dots, a_{x_{n-1} x_{n-2}}, a_{x_n x_{n-1}}$, to vertex x_n . Then $\omega(\alpha) = \alpha \cdot x_0^{-1}$.

(ii) Let $v = \sum_{p=1}^m k_p \alpha_p \in P(R)$, where $\alpha_p = b_{x_n x_{n-1}}^{(p)} b_{x_{n-1} x_{n-2}}^{(p)} \cdots b_{x_1 x_0}^{(p)}$ is a path with $n > 1$, $k_p \in k$ for $p = 1, 2, \dots, m$, and $b_{x_j x_{j-1}}^{(p)}$ is an arrow from vertex x_{j-1} to vertex x_j for $j = 1, 2, \dots, n$. We shall show that $k_p = 0$ for $p = 1, 2, \dots, m$. Indeed,

$$\begin{aligned} \Delta_R(v) &= \sum_{p=1}^m \sum_{j=0}^n k_p (b_{x_n x_{n-1}}^{(p)} b_{x_{n-1} x_{n-2}}^{(p)} \cdots b_{x_{j+1} x_j}^{(p)}) \cdot (x_j^{(p)})^{-1} \otimes b_{x_j x_{j-1}}^{(p)} b_{x_{j-1} x_{j-2}}^{(p)} \cdots b_{x_1 x_0}^{(p)} \\ &= \sum_{p=1}^m k_p (\alpha_p \otimes 1 + 1 \otimes \alpha_p). \end{aligned}$$

This implies

$$\sum_{p=1}^m k_p (b_{x_n x_{n-1}}^{(p)} b_{x_{n-1} x_{n-2}}^{(p)} \cdots b_{x_{j+1} x_j}^{(p)}) \cdot (x_j^{(p)})^{-1} \otimes b_{x_j x_{j-1}}^{(p)} b_{x_{j-1} x_{j-2}}^{(p)} \cdots b_{x_1 x_0}^{(p)} = 0, \quad (2.1)$$

for $j = 1, 2, \dots, n-1$, because of their length. For any j with $1 \leq j \leq n-1$, assume that $\{p \mid b_{x_j x_{j-1}}^{(p)} b_{x_{j-1} x_{j-2}}^{(p)} \cdots b_{x_1 x_0}^{(p)} = b_{x_j x_{j-1}}^{(1)} b_{x_{j-1} x_{j-2}}^{(1)} \cdots b_{x_1 x_0}^{(1)}\} = \{1, 2, \dots, m_1\}$ without loss generality. Therefore, by (2.1),

$$\sum_{p=1}^{m_1} k_p (b_{x_n x_{n-1}}^{(p)} b_{x_{n-1} x_{n-2}}^{(p)} \cdots b_{x_{j+1} x_j}^{(p)}) \cdot (x_j^{(p)})^{-1} \otimes b_{x_j x_{j-1}}^{(p)} b_{x_{j-1} x_{j-2}}^{(p)} \cdots b_{x_1 x_0}^{(p)} = 0,$$

since $b_{x_j x_{j-1}}^{(q)} b_{x_{j-1} x_{j-2}}^{(q)} \cdots b_{x_1 x_0}^{(q)} \neq b_{x_j x_{j-1}}^{(1)} b_{x_{j-1} x_{j-2}}^{(1)} \cdots b_{x_1 x_0}^{(1)}$ for $q = m_1 + 1, \dots, m$. This implies $\sum_{p=1}^{m_1} k_p b_{x_n x_{n-1}}^{(p)} b_{x_{n-1} x_{n-2}}^{(p)} \cdots b_{x_{j+1} x_j}^{(p)} = 0$ and $k_p = 0$ for $p = 1, 2, \dots, m_1$. Similarly, we can show $k_p = 0$ for $p = m_1 + 1, \dots, m$. \square

By [BD, Theorem 4.3.2], the category ${}^H_H \mathcal{YD}$ of Yetter-Drinfeld modules is equivalent to the category ${}^H_H \mathcal{M}_H^H$ of H -Hopf bimodule, where H is a Hopf algebra with bijective antipode. Let T and U be the two corresponding functors. For any $N \in {}^H_H \mathcal{YD}$, according to [BD, Prop.4.2.1], $T(N) := N \rtimes H = N \otimes H$ as vector spaces, and the actions and coactions are given as follows: the left (co)actions are diagonal and right (co)actions are induced by H . Explicitly, $g \cdot (x \otimes h) := g \cdot x \otimes gh$, $(x \otimes h) \cdot g = x \otimes hg$, $\delta_{N \rtimes H}^-(x \otimes h) := \sum_x x_{(-1)} h \otimes x_{(0)} \otimes h$; $\delta_{N \rtimes H}^+(x \otimes h) := x \otimes h \otimes h$, where $\delta_N^-(x) = \sum_x x_{(-1)} \otimes x_{(0)}$, $x \in N, h, g \in H$. For any $M \in {}^H_H \mathcal{M}_H^H$, according to [BD, Equ. (7) and (21)], $U(M)$ is the coinvariant of M as a vector space, i.e., $U(M) := M^{\text{co}H} := \{x \in M \mid \delta_N^+(x) = x \otimes 1\}$. The left action is left adjoint action and the left coaction is the restricted coaction of the original coaction of M . That is,

$$\alpha_{U(M)}^-(h \otimes x) = h \triangleright_{\text{ad}} x := \alpha_M^+(\alpha_M^-(h \otimes x) \otimes h^{-1}) = (h \cdot x) \cdot h^{-1}$$

and $\delta_{U(M)}^-(x) = \delta_M^-(x)$ for any $h \in H, x \in U(M)$. In fact, $TU(M) = U(M) \rtimes H$ and $UT(N) = N \otimes 1_H$. Let λ_N be a map from $N \otimes 1_H$ to N by sending $x \otimes 1_H$ to x for any $x \in N$, and let ν_M be a map from $U(M) \rtimes H$ to M by sending $x \otimes h$ to $\alpha_M^+(x \otimes h) = x \cdot h$ for any $x \in U(M)$ and $h \in H$. λ and ν are the natural isomorphisms from functor UT to id and from functor TU to id , respectively. Note that the inverse of ν_M is $(\alpha_M^+ \otimes id)(id \otimes S \otimes id)(\delta_M^+ \otimes id)\delta_M^+$.

Remark: We have $U(kQ_1^c, G, r, \vec{\rho}, u) = (kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ by the proof of Lemma 2.2.

Lemma 2.3. Assume that ϕ is a Hopf algebra isomorphism from H to H' . Let $N \in {}^{H'}_{H'} \mathcal{YD}$ and $M \in {}^{H'}_{H'} \mathcal{M}_{H'}^{H'}$. Then

$$T(\phi^{-1}N) \cong \phi^{-1}T(N)\phi^{-1} \text{ in } {}^H_H \mathcal{M}_H^H \text{ and } U(\phi^{-1}M\phi^{-1}) \cong \phi^{-1}U(M) \text{ in } {}^H_H \mathcal{YD}.$$

Proof. The first isomorphism is given by sending $x \otimes h$ to $x \otimes \phi(h)$ for any $x \in N, h \in H$; the second one is identity. \square

Proposition 2.4. (i) If N is a Yetter-Drinfeld kG -module, then there exists an $\text{RSR}(G, r, \vec{\rho}, u)$ such that $N \cong (kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ as Yetter-Drinfeld kG -modules.

(ii) If $\mathcal{B}(N)$ is a Nichols algebra in ${}^{kG}_{kG} \mathcal{YD}$, then there exists an $\text{RSR}(G, r, \vec{\rho}, u)$ such that $\mathcal{B}(N) \cong \mathcal{B}(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ as graded braided Hopf algebras in ${}^{kG}_{kG} \mathcal{YD}$.

Proof. (i) Since $T(N)$ is a kG -Hopf bimodule, it follows from Proposition 1.3 that there exists an $\text{RSR}(G, r, \vec{\rho}, u)$ such that $T(N) \cong (kQ_1^c, G, r, \vec{\rho}, u)$ as kG -Hopf bimodules.

Thus, $N \cong UT(N) \cong U(kQ_1^c, G, r, \vec{\rho}, u) = (kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ as Yetter-Drinfeld kG -modules by [BD, Equ. (7) and (21)].

(ii) It follows from (i) and [AS02, Cor.2.3]. \square

Lemma 2.5. *Assume that ϕ is a Hopf algebra isomorphism from H to H' . Let R and R' be graded braided Hopf algebras in ${}^H_H\mathcal{YD}$ and ${}^{H'}_{H'}\mathcal{YD}$ with $R_{(0)} = k1_R$ and $R'_{(0)} = k1_{R'}$, respectively. If R and $\phi^{-1}R'$ are isomorphic as graded braided Hopf algebras in ${}^H_H\mathcal{YD}$, then biproducts $R\#H \cong R'\#H'$ as graded Hopf algebras.*

Proof. Let ψ be a graded braided Hopf algebra isomorphism from R to $\phi^{-1}R'$ in ${}^H_H\mathcal{YD}$. Define a map ν from $R\#H$ to $R'\#H'$ by sending $x \otimes h$ to $\psi(x) \otimes \phi(h)$ for any $x \in R, h \in H$. It is easy to check that ν is an isomorphism of graded Hopf algebras. \square

Theorem 3. *If A is a pointed Hopf algebra of Nichols type with coradical kG , a group algebra, then there exists a unique $\text{RSR}(G, r, \vec{\rho}, u)$, up to isomorphism, such that $A \cong kG[kQ_1^c, G, r, \vec{\rho}, u]$ as graded Hopf algebras.*

Proof. By Lemma 2.1, $A \cong \mathcal{B}(V)\#kG$ as graded Hopf algebras. By Proposition 2.4 (ii), there exists a $\text{RSR}(G, r, \vec{\rho}, u)$ such that $\mathcal{B}(V) \cong \mathcal{B}(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ as graded braided Hopf algebras in ${}^{kG}_{kG}\mathcal{YD}$. Thus

$$\begin{aligned} kG[kQ_1^c, G, r, \vec{\rho}, u] &\cong \mathcal{B}(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))\#kG \quad (\text{by Lemma 2.2}) \\ &\cong \mathcal{B}(V)\#kG \quad (\text{by Lemma 2.5}) \\ &\cong A. \end{aligned}$$

The uniqueness follows from [ZZC, Theorem 3]. \square

Considering Theorem 3 and Lemma 2.2, we have that A is a pointed Hopf algebra of Nichols type if and only if A is a pointed Hopf algebra of type one.

3 Classification of Nichols Algebras

Theorem 4. *Let $(G, r, \vec{\rho}, u)$ and $(G', r', \vec{\rho}', u')$ be two RSR's. Then the following statements are equivalent:*

- (i) $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G', r', \vec{\rho}', u')$.
- (ii) *There exists a Hopf algebra isomorphism $\phi : kG \rightarrow kG'$ such that $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u)) \cong \phi^{-1}(kQ_1^1, \text{ad}(G', r', \vec{\rho}', u'))$ as Yetter-Drinfeld kG -modules.*
- (iii) *There is a Hopf algebra isomorphism $\phi : kG \rightarrow kG'$ such that $\mathcal{B}(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u)) \cong \phi^{-1}\mathcal{B}(kQ_1^1, \text{ad}(G', r', \vec{\rho}', u'))$ as graded braided Hopf algebra in ${}^{kG}_{kG}\mathcal{YD}$.*

Proof. (i) \Rightarrow (ii). See

$$\begin{aligned}
\phi^{-1}(kQ_1'^1, \text{ad}(G', r', \vec{\rho}', u')) &= \phi^{-1}U(kQ_1'^c, G', r', \vec{\rho}', u') \quad (\text{by the remark before Lemma 2.3}) \\
&\cong U(\phi^{-1}(kQ_1'^c, G', r', \vec{\rho}', u'))_{\phi}^{\phi^{-1}} \quad (\text{by Lemma 2.3}) \\
&\cong U((kQ_1^c, G, r, \vec{\rho}, u)) \quad (\text{by Theorem 2}) \\
&= (kQ_1^1, \text{ad}(G, r, \vec{\rho}, u)).
\end{aligned}$$

(ii) \Rightarrow (iii). By [ZZC, Lemma 2.7], $\mathcal{B}(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u)) \cong \mathcal{B}(\phi^{-1}(kQ_1'^1, \text{ad}(G', r', \vec{\rho}', u')))$
 $\cong \phi^{-1}\mathcal{B}(kQ_1'^1, \text{ad}(G', r', \vec{\rho}', u'))$ as graded braided Hopf algebras in ${}^kG\mathcal{YD}$.

(iii) \Rightarrow (i). $\mathcal{B}(kQ_1^1)\#kG \cong \mathcal{B}(kQ_1'^1)\#kG'$ as graded Hopf algebras by Lemma 2.5.
Thus $kG[kQ_1^c] \cong \mathcal{B}(kQ_1^1)\#kG \cong \mathcal{B}(kQ_1'^1)\#kG' \cong kG'[kQ_1'^c]$ as graded Hopf algebras by Lemma 2.2. Now (i) follows from Theorem 2. \square

Up to now we have classified all Nichols algebras by means of RSR's. In other words, ramification systems with irreducible representations uniquely determine their corresponding Nichols algebras up to pull-push graded braided Hopf algebra isomorphisms.

4 The classification of RSR over symmetric groups

Let ad_h^- and ad_h^+ denote the left and right adjoint actions, respectively. That is, $\text{ad}_h^-(x) := h x h^{-1}$ for any $x \in G$. Let $\gamma_C = |Z_s|$ when $s \in C \in \mathcal{K}(G)$. $\text{Aut}G$ and $\text{Inn}G$ denote the automorphism group and inner automorphism group of G .

Definition 4.1. Let $\text{RSR}(G, r, \vec{\rho}, u)$ and $\text{RSR}(G', r', \vec{\rho}', u')$ be two RSR's. If there exists a bijective map ϕ_C from $I_C(r, u)$ to $I_C(r', u')$ such that $\rho_C^{(i)} \cong \rho_C'^{(\phi_C(i))}$ for any $i \in I_C(r, u)$ and $C \in \mathcal{K}_r(G)$ with $G = G'$, $r = r'$ and $u = u'$, then $\text{RSR}(G, r, \vec{\rho}, u)$ and $\text{RSR}(G', r', \vec{\rho}', u')$ are said to be of the same type. Furthermore, if let $\widehat{Z_{u(C)}} = \{\xi_{u(C)}^{(i)} \mid i = 1, 2, \dots, \gamma_C\}$ and $n_C^{(i)} := |\{j \mid \rho_C^{(j)} \cong \xi_{u(C)}^{(i)}\}|$ for any $C \in \mathcal{K}_r(G)$ and $1 \leq i \leq \gamma_{u(C)}$, then $\{(n_C^{(1)}, n_C^{(2)}, \dots, n_C^{(\gamma_C)})\}_{C \in \mathcal{K}_r(G)}$ is called the type of $\text{RSR}(G, r, \vec{\rho}, u)$.

Lemma 4.2. If $\text{Aut}G = \text{Inn}G$, for example, $G = \mathbb{S}_n$ with $n \neq 6$, then $\text{RSR}(G, r, \vec{\rho}, u)$ and $\text{RSR}(G, r, \vec{\rho}', u)$ are isomorphic if and only if they have the same type.

Proof. By [Zh82, Theorem 1.12.7] or [ZWW, Proposition 1.1 (ii)], $\text{Aut}G = \text{Inn}G$ when $G = \mathbb{S}_n$ with $n \neq 6$.

Let $\text{RSR}(G', r', \vec{\rho}', u')$ denote $\text{RSR}(G, r, \vec{\rho}', u)$ with $G = G'$, $r = r'$ and $u = u'$ for convenience. If $\text{RSR}(G, r, \vec{\rho}, u)$ and $\text{RSR}(G, r, \vec{\rho}', u)$ have the same type, then there exists a bijective map ϕ_C from $I_C(r, u)$ to $I_{\phi(C)}(r', u')$ such that $\rho_C'^{(\phi_C(i))} \cong \rho_C^{(i)}$ for any $i \in I_C(r, u)$, $C \in \mathcal{K}_r(G)$. Therefore they are isomorphic.

Conversely, if $\text{RSR}(G, r, \vec{\rho}, u)$ and $\text{RSR}(G, r, \vec{\rho}', u)$ are isomorphic, then there exist a $\phi \in \text{Aut}(G)$, $h_C \in G$ and a bijective map $\phi_C : I_C(r, u) \rightarrow I_{\phi(C)}(r', u')$ such that $u'(\phi(C)) = \phi \text{ad}_{h_C}^+(u(C))$ and $\rho'_{\phi(C)}^{(\phi_C(i))} \phi \text{ad}_{h_C}^+ \cong \rho_C^{(i)}$ for any $C \in \mathcal{K}_r(G)$ and $i \in I_C(r, u)$. Since $\phi \text{ad}_{h_C}^+ \in \text{Aut}G = \text{Inn}G$, there exists a $g_C \in G$ such that $\phi \text{ad}_{h_C}^+ = \text{ad}_{g_C}^+$. Therefore $u(C) = \text{ad}_{g_C}^+(u(C))$ and $\rho_C'^{(\phi_C(i))} \text{ad}_{g_C}^+ \cong \rho_C^{(i)}$. That is, $g_C \in Z_{u(C)}$ and $\chi_C'^{(\phi_C(i))} \text{ad}_{g_C}^+(h) = \chi_C'^{(\phi_C(i))}(g_C^{-1} h g_C) = \chi_C'^{(\phi_C(i))}(h) = \chi_C^{(i)}(h)$ for any $h \in Z_{u(C)}$, where $\chi_C'^{(\phi_C(i))}$ and $\chi_C^{(i)}$ denote the characters of $\rho_C'^{(\phi_C(i))}$ and $\rho_C^{(i)}$, respectively. Consequently, $\chi_C'^{(\phi_C(i))} = \chi_C^{(i)}$ and $\rho_C'^{(\phi_C(i))} \cong \rho_C^{(i)}$. This implies that $\text{RSR}(G, r, \vec{\rho}, u)$ and $\text{RSR}(G, r, \vec{\rho}', u)$ have the same type. \square

For a given ramification r of G , let $\Omega(G, r)$ be the set of all RSR's of G with the ramification r , namely, $\Omega(G, r) := \{(G, r, \vec{\rho}, u) \mid (G, r, \vec{\rho}, u) \text{ is an RSR}\}$. Let $\mathcal{N}(G, r)$ be the number of isomorphic classes in $\Omega(G, r)$.

Theorem 5. *Given a group G and a ramification r of G . Assume $\text{Aut}G = \text{Inn}G$, for example, $G = \mathbb{S}_n$ with $n \neq 6$. Let u_0 be a fixed map from $\mathcal{K}(G) \rightarrow G$ with $u_0(C) \in C$ for any $C \in \mathcal{K}(G)$. Let $\bar{\Omega}(G, r, u_0)$ denote the set consisting of all elements with distinct type in $\{(G, r, \vec{\rho}, u_0) \mid (G, r, \vec{\rho}, u_0) \text{ is an RSR}\}$. Then $\bar{\Omega}(G, r, u_0)$ becomes the representative system of isomorphic classes in $\Omega(G, r)$.*

Proof. For any $\text{RSR}(G, r, \vec{\rho}, u)$, there exists $\text{RSR}(G, r, \vec{\rho}', u_0)$ such that $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G, r, \vec{\rho}', u_0)$ by Proposition 1.5. Using Lemma 4.2, we complete the proof. \square

Corollary 4.3.

$$\mathcal{N}(G, r) = \prod_{C \in \mathcal{K}_r(G)} \tau_C,$$

where $\tau_C =$ the number of elements in set $\{(n_1, n_2, \dots, n_{\gamma_C}) \in \mathbb{N}^{\gamma_C} \mid n_1 \deg \xi_1 + n_2 \deg \xi_2 + \dots + n_{\gamma_C} \deg \xi_{\gamma_C} = r_C\}$ for any $C \in \mathcal{K}_r(G)$.

Remark: For a given finite Hopf quiver (Q, G, r) over symmetric group $G = \mathbb{S}_n$ with $n \neq 6$, every path algebra $T_{(kG)^*}(kQ_1^a)$ over quiver (Q, G, r) exactly admits $\mathcal{N}(G, r)$ non-isomorphic graded Hopf algebra structures; every path coalgebra $T_{kG}^c(kQ_1^c)$ over quiver (Q, G, r) exactly admits $\mathcal{N}(G, r)$ non-isomorphic graded Hopf algebra structures.

5 Appendix

We now consider the dual case of Theorem 4. If Q is finite, then $(kQ_1^a, G, r, \vec{\rho}, u)$ is a $(kG)^*$ -Hopf bimodule with comodule operations δ^- and δ^+ . Define a new left $(kG)^*$ -coaction on kQ_1^a given by

$$\delta_{\text{coad}}^-(x) := \sum_x x_{(-1)} S(x_{(0)(1)}) \otimes x_{(0)(0)}, \quad \text{for any } x \in kQ_1^a,$$

i.e. adjoint coaction. With this left $(kG)^*$ -coaction and the original left (arrow) $(kG)^*$ -action α^- , kQ_1^a is a Yetter-Drinfeld $(kG)^*$ -module. Let kQ_1^{1a} denote the subspace spanned by Q_1^1 in kQ_1^a . It is clear that $\xi_{kQ_1^c}(kQ_1^1) = (kQ_1^{1a})^*$, where $\xi_{kQ_1^c}$ was defined in [ZZC, Lemma 1.7]. Thus kQ_1^{1a} is a Yetter-Drinfeld $(kG)^*$ -submodule of kQ_1^a , denoted by $(kQ_1^{1a}, \text{coad}(G, r, \vec{\rho}, u))$, which is isomorphic to the dual of $(kQ_1^1, \text{ad}(G, r, \vec{\rho}, u))$ as Yetter-Drinfeld $(kG)^*$ -modules.

Therefore we have the dual case of Theorem 4.

Proposition 5.1. *Let $(G, r, \vec{\rho}, u)$ and $(G', r', \vec{\rho}', u')$ be two RSR's. Then the following statements are equivalent:*

- (i) $\text{RSR}(G, r, \vec{\rho}, u) \cong \text{RSR}(G', r', \vec{\rho}', u')$.
- (ii) *There exists a Hopf algebra isomorphism $\phi : (kG)^* \rightarrow (kG')^*$ such that $(kQ_1^{1a}, \text{coad}(G, r, \vec{\rho}, u)) \cong_{\phi^{-1}} (kQ_1'^{1a}, \text{coad}(G', r', \vec{\rho}', u'))$ as Yetter-Drinfeld $(kG)^*$ -modules.*
- (iii) *There is a Hopf algebra isomorphism $\phi : (kG)^* \rightarrow (kG')^*$ such that $\mathcal{B}(kQ_1^{1a}, \text{coad}(G, r, \vec{\rho}, u)) \cong_{\phi^{-1}} \mathcal{B}(kQ_1'^{1a}, \text{coad}(G', r', \vec{\rho}', u'))$ as graded braided Hopf algebras in ${}^{(kG)^*}_{(kG)^*}\mathcal{YD}$.*

Proof. Obviously, (ii) and Theorem 4 (ii) are equivalent; (iii) and Theorem 4 (iii) are equivalent. \square

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